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Sinc function

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"Sinc" redirects here. For the designation used in the United Kingdom for areas of wildlife interest, see [Site of Importance for Nature Conservation](#). For the signal processing filter based on this function, see [Sinc filter](#).

In [mathematics](#), [physics](#) and [engineering](#), the **cardinal sine function** or **sinc function**, denoted by $\operatorname{sinc}(x)$, has two slightly different definitions.^[1]

In mathematics, the historical **unnormalized sinc function** is defined for $x \neq 0$ by

$$\operatorname{sinc}(x) = \frac{\sin(x)}{x}.$$

In [digital signal processing](#) and [information theory](#), the **normalized sinc function** is commonly defined for $x \neq 0$ by

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}.$$

In either case, the value at $x = 0$ is defined to be the limiting value

$$\operatorname{sinc}(0) := \lim_{x \rightarrow 0} \frac{\sin(ax)}{ax} = 1 \text{ for all real } a \neq 0.$$

The [normalization](#) causes the [definite integral](#) of the function over the real numbers to equal 1 (whereas the same integral of the unnormalized sinc function has a value of π). As a further useful property, the zeros of the normalized sinc function are the nonzero integer values of x .

The normalized sinc function is the [Fourier transform](#) of the [rectangular function](#) with no scaling. It is used in the concept of [reconstructing](#) a continuous bandlimited signal from uniformly spaced [samples](#) of that signal.



The only difference between the two definitions is in the scaling of the **independent variable** (the *x*-axis) by a factor of π . In both cases, the value of the function at the **removable singularity** at zero is understood to be the limit value 1. The sinc function is then **analytic** everywhere and hence an **entire function**.

The term *sinc* /ˈsɪnk/ is a contraction of the function's full Latin name, the *sinus cardinalis* (cardinal sine).^[2] It was introduced by Philip M. Woodward in his 1952 paper "Information theory and inverse probability in telecommunication", in which he said the function "occurs so often in Fourier analysis and its applications that it does seem to merit some notation of its own",^[3] and his 1953 book *Probability and Information Theory, with Applications to Radar*.^{[2][4]}

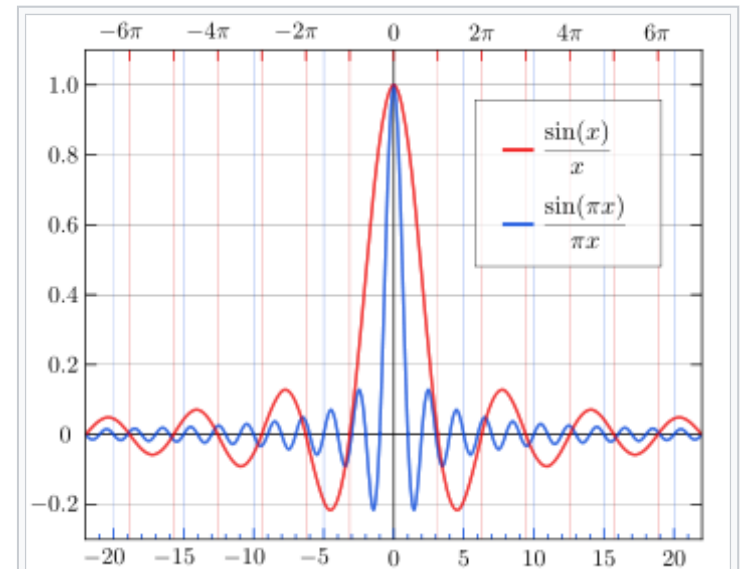
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Properties [edit]

The **zero crossings** of the unnormalized sinc are at non-zero integer multiples of π , while zero crossings of the normalized sinc occur at non-zero integers.

The local maxima and minima of the unnormalized sinc correspond to its intersections with the cosine function. That is, $\frac{\sin(\zeta)}{\zeta} = \cos(\zeta)$ for all points ζ where the derivative of $\frac{\sin(x)}{x}$ is zero and thus a local extremum is reached. This follows from the derivative of the sinc



The normalized sinc (blue) and unnormalized sinc function (red) shown on the same scale.

function,

$$\frac{d \operatorname{sinc}(x)}{dx} = \frac{\cos(x) - \operatorname{sinc}(x)}{x}$$

The first few terms of the infinite series for the x -coordinate of the n th extremum with positive x -coordinate are

$$x_n = q - q^{-1} - \frac{2}{3}q^{-3} - \frac{13}{15}q^{-5} - \frac{146}{105}q^{-7} - \dots$$

where

$$q = \left(n + \frac{1}{2}\right) \pi$$

and where odd n lead to a local minimum and even n to a local maximum. Because of symmetry around the y -axis, there exist extrema with x -coordinates $-x_n$. In addition, there is an absolute maximum at $\zeta_0 = (0, 1)$.

The normalized sinc function has a simple representation as the [infinite product](#)

$$\frac{\sin(\pi x)}{\pi x} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right)$$

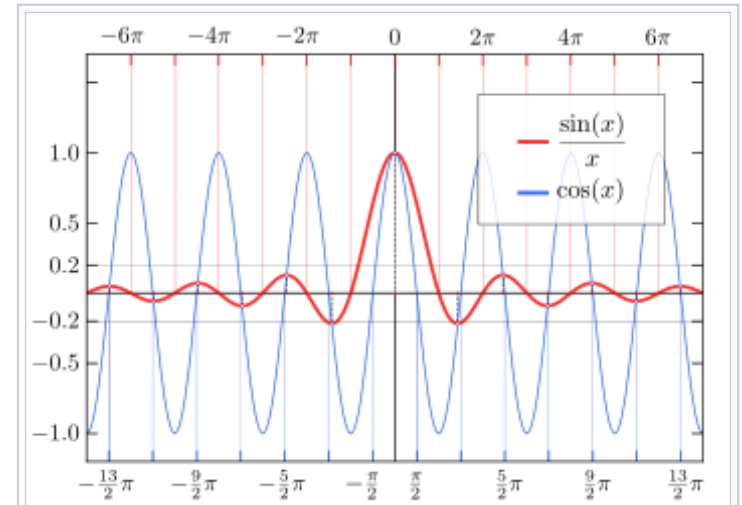
and is related to the [gamma function](#) $\Gamma(x)$ through [Euler's reflection formula](#),

$$\frac{\sin(\pi x)}{\pi x} = \frac{1}{\Gamma(1+x)\Gamma(1-x)}.$$

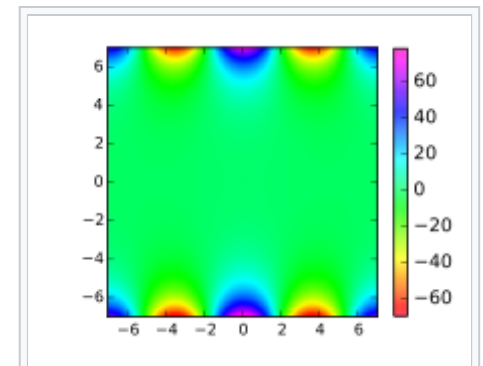
[Euler](#) discovered^[5] that

$$\frac{\sin(x)}{x} = \prod_{n=1}^{\infty} \cos\left(\frac{x}{2^n}\right)$$

and because of the product-to-sum identity^[6]



The local maxima and minima (small white dots) of the unnormalized, red sinc function correspond to its intersections with the blue [cosine function](#).



The real part of complex sinc $\operatorname{Re}(\operatorname{sinc} z) = \operatorname{Re}\left(\frac{\sin z}{z}\right)$.

$$\prod_{n=1}^k \cos\left(\frac{x}{2^n}\right) = \frac{1}{2^{k-1}} \sum_{n=1}^{2^{k-1}} \cos\left(\frac{n-1/2}{2^{k-1}}x\right), \quad \forall k \geq 1,$$

the Euler's product can be recast as a sum

$$\frac{\sin(x)}{x} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \cos\left(\frac{n-1/2}{N}x\right).$$

The [continuous Fourier transform](#) of the normalized sinc (to ordinary frequency) is [rect\(f\)](#),

$$\int_{-\infty}^{\infty} \text{sinc}(t) e^{-i2\pi ft} dt = \text{rect}(f),$$

where the [rectangular function](#) is 1 for argument between $-\frac{1}{2}$ and $\frac{1}{2}$, and zero otherwise. This corresponds to the fact that the [sinc filter](#) is the ideal ([brick-wall](#), meaning rectangular frequency response) [low-pass filter](#).

This Fourier integral, including the special case

$$\int_{-\infty}^{\infty} \frac{\sin(\pi x)}{\pi x} dx = \text{rect}(0) = 1$$

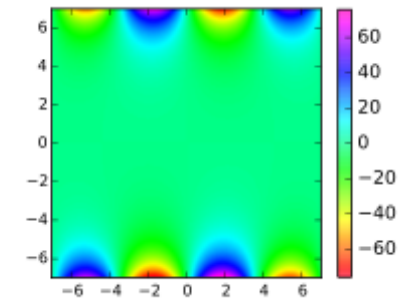
is an [improper integral](#) (cf. [Dirichlet integral](#)) and not a convergent [Lebesgue integral](#), as

$$\int_{-\infty}^{\infty} \left| \frac{\sin(\pi x)}{\pi x} \right| dx = +\infty.$$

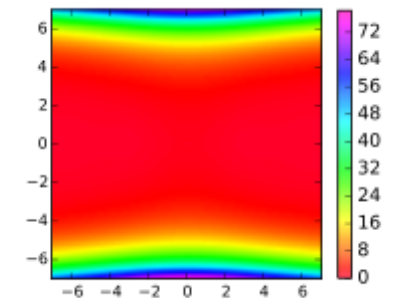
The normalized sinc function has properties that make it ideal in relationship to [interpolation](#) of [sampled bandlimited](#) functions:

- It is an interpolating function, i.e., $\text{sinc}(0) = 1$, and $\text{sinc}(k) = 0$ for nonzero [integer](#) k .
- The functions $x_k(t) = \text{sinc}(t - k)$ (k integer) form an [orthonormal basis](#) for [bandlimited](#) functions in the [function space](#) $L^2(\mathbf{R})$, with highest angular frequency $\omega_H = \pi$ (that is, highest cycle frequency $f_H = \frac{1}{2}$).

Other properties of the two sinc functions include:



The imaginary part of complex sinc $\Im(\text{sinc } z) = \Im\left(\frac{\sin z}{z}\right)$.



The absolute value $|\text{sinc } z| = \left| \frac{\sin z}{z} \right|$.

- The unnormalized sinc is the zeroth-order spherical [Bessel function](#) of the first kind, $j_0(x)$. The normalized sinc is $j_0(\pi x)$.

- $$\int_0^x \frac{\sin(\theta)}{\theta} d\theta = \text{Si}(x)$$

where $\text{Si}(x)$ is the [sine integral](#).

- $\lambda \text{sinc}(\lambda x)$ (not normalized) is one of two linearly independent solutions to the linear [ordinary differential equation](#)

$$x \frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} + \lambda^2 xy = 0.$$

The other is $\frac{\cos(\lambda x)}{x}$, which is not bounded at $x = 0$, unlike its sinc function counterpart.

- $$\int_{-\infty}^{\infty} \frac{\sin^2(\theta)}{\theta^2} d\theta = \pi \rightarrow \int_{-\infty}^{\infty} \text{sinc}^2(x) dx = 1,$$

where the normalized sinc is meant.

- $$\int_{-\infty}^{\infty} \frac{\sin(\theta)}{\theta} d\theta = \int_{-\infty}^{\infty} \left(\frac{\sin(\theta)}{\theta} \right)^2 d\theta = \pi$$

- $$\int_{-\infty}^{\infty} \frac{\sin^3(\theta)}{\theta^3} d\theta = \frac{3\pi}{4}$$

- $$\int_{-\infty}^{\infty} \frac{\sin^4(\theta)}{\theta^4} d\theta = \frac{2\pi}{3}.$$

- The following improper integral involves the (not normalized) sinc function:

- $$\int_0^{\infty} \frac{dx}{x^n + 1} = 1 + 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(kn)^2 - 1} = \frac{1}{\text{sinc}(\frac{\pi}{n})}$$

Relationship to the Dirac delta distribution [\[edit \]](#)

The normalized sinc function can be used as a [nascent delta function](#), meaning that the following [weak limit](#) holds,

$$\lim_{a \rightarrow 0} \frac{\sin\left(\frac{\pi x}{a}\right)}{\pi x} = \lim_{a \rightarrow 0} \frac{1}{a} \operatorname{sinc}\left(\frac{x}{a}\right) = \delta(x) .$$

This is not an ordinary limit, since the left side does not converge. Rather, it means that

$$\lim_{a \rightarrow 0} \int_{-\infty}^{\infty} \frac{1}{a} \operatorname{sinc}\left(\frac{x}{a}\right) \varphi(x) dx = \varphi(0) ,$$

for every [Schwartz function](#), as can be seen from the [Fourier inversion theorem](#). In the above expression, as $a \rightarrow 0$, the number of oscillations per unit length of the sinc function approaches infinity. Nevertheless, the expression always oscillates inside an envelope of $\pm \frac{1}{\pi x}$, regardless of the value of a .

This complicates the informal picture of $\delta(x)$ as being zero for all x except at the point $x = 0$, and illustrates the problem of thinking of the delta function as a function rather than as a distribution. A similar situation is found in the [Gibbs phenomenon](#).

Summation [\[edit \]](#)

All sums in this section refer to the unnormalized sinc function.

The sum of $\operatorname{sinc}(n)$ over integer n from 1 to ∞ equals $\frac{\pi-1}{2}$.

$$\sum_{n=1}^{\infty} \operatorname{sinc}(n) = \operatorname{sinc}(1) + \operatorname{sinc}(2) + \operatorname{sinc}(3) + \operatorname{sinc}(4) + \cdots = \frac{\pi-1}{2}$$

The sum of the squares also equals $\frac{\pi-1}{2}$.^{[\[7\]](#)}

$$\sum_{n=1}^{\infty} \operatorname{sinc}^2(n) = \operatorname{sinc}^2(1) + \operatorname{sinc}^2(2) + \operatorname{sinc}^2(3) + \operatorname{sinc}^2(4) + \cdots = \frac{\pi-1}{2}$$

When the signs of the [addends](#) alternate and begin with +, the sum equals $\frac{1}{2}$.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \operatorname{sinc}(n) = \operatorname{sinc}(1) - \operatorname{sinc}(2) + \operatorname{sinc}(3) - \operatorname{sinc}(4) + \cdots = \frac{1}{2}$$

The alternating sums of the squares and cubes also equal $\frac{1}{2}$.^{[\[8\]](#)}

$$\sum_{n=1}^{\infty} (-1)^{n+1} \operatorname{sinc}^2(n) = \operatorname{sinc}^2(1) - \operatorname{sinc}^2(2) + \operatorname{sinc}^2(3) - \operatorname{sinc}^2(4) + \cdots = \frac{1}{2}$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \operatorname{sinc}^3(n) = \operatorname{sinc}^3(1) - \operatorname{sinc}^3(2) + \operatorname{sinc}^3(3) - \operatorname{sinc}^3(4) + \cdots = \frac{1}{2}$$

Series expansion [\[edit \]](#)

Unnormalized $\operatorname{sinc}(x)$:

$$\operatorname{sinc}(x) = \frac{\sin(x)}{x} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{(2n+1)!}$$

Higher dimensions [\[edit \]](#)

The product of 1-D sinc functions readily provides a [multivariate](#) sinc function for the square, Cartesian, grid ([lattice](#)):

$\operatorname{sinc}_C(x, y) = \operatorname{sinc}(x)\operatorname{sinc}(y)$ whose [Fourier transform](#) is the [indicator function](#) of a square in the frequency space (i.e., the brick wall defined in 2-D space). The sinc function for a non-Cartesian [lattice](#) (e.g., [hexagonal lattice](#)) is a function whose [Fourier transform](#) is the [indicator function](#) of the [Brillouin zone](#) of that lattice. For example, the sinc function for the hexagonal lattice is a function whose [Fourier transform](#) is the [indicator function](#) of the unit hexagon in the frequency space. For a non-Cartesian lattice this function can not be obtained by a simple tensor-product. However, the explicit formula for the sinc function for the [hexagonal](#), [body centered cubic](#), [face centered cubic](#) and other higher-dimensional lattices can be explicitly derived^[9] using the geometric properties of [Brillouin zones](#) and their connection to [zonotopes](#).

For example, a [hexagonal lattice](#) can be generated by the (integer) [linear span](#) of the vectors

$$\mathbf{u}_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix}.$$

Denoting

$$\boldsymbol{\xi}_1 = \frac{2}{3}\mathbf{u}_1, \quad \boldsymbol{\xi}_2 = \frac{2}{3}\mathbf{u}_2, \quad \boldsymbol{\xi}_3 = -\frac{2}{3}(\mathbf{u}_1 + \mathbf{u}_2), \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix},$$

one can derive^[9] the sinc function for this hexagonal lattice as:

$$\begin{aligned} \text{sinc}_H(\mathbf{x}) = \frac{1}{3} & \left(\cos(\pi\boldsymbol{\xi}_1 \cdot \mathbf{x}) \text{sinc}(\boldsymbol{\xi}_2 \cdot \mathbf{x}) \text{sinc}(\boldsymbol{\xi}_3 \cdot \mathbf{x}) \right. \\ & + \cos(\pi\boldsymbol{\xi}_2 \cdot \mathbf{x}) \text{sinc}(\boldsymbol{\xi}_3 \cdot \mathbf{x}) \text{sinc}(\boldsymbol{\xi}_1 \cdot \mathbf{x}) \\ & \left. + \cos(\pi\boldsymbol{\xi}_3 \cdot \mathbf{x}) \text{sinc}(\boldsymbol{\xi}_1 \cdot \mathbf{x}) \text{sinc}(\boldsymbol{\xi}_2 \cdot \mathbf{x}) \right) \end{aligned}$$










This construction can be used to design [Lanczos window](#) for general multidimensional lattices.^[9]

See also [[edit](#)]

- [Anti-aliasing filter](#)
- [Sinc filter](#)
- [Lanczos resampling](#)
- [Whittaker–Shannon interpolation formula](#)
- [Shannon wavelet](#)
- [Winkel tripel projection](#) (cartography)
- [Trigonometric integral](#)
- [Trigonometric functions of matrices](#)
- [Borwein integral](#)
- [Dirichlet integral](#)

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External links

-  Weisstein, Eric W. "Sinc Function" . *MathWorld*.

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